

Some Basic Probability Notation and Terminology

Let A and B be two sets.

1. Suppose every element of a set A also belongs to set B, then A is called a **subset** of B, or A is said to be contained in B, which is written as

$$\mathbf{A \subseteq B \text{ or } B \supseteq A}$$

2. Two sets are **equal** if they both have the same elements or, equivalently, if each is contained in the other.

$$\mathbf{A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A}$$

3. The statement that an element *a* belongs to a set S is written

$$\mathbf{a \in S}$$

Here \in is the symbol meaning “is an element of”.

4. The **negation** of

$$\mathbf{a \in S, A \subseteq B \text{ and } A = B}$$

are written

$$\mathbf{a \notin S, A \not\subseteq B \text{ and } A \neq B}$$

5. If $\mathbf{A \subseteq B}$ and $\mathbf{A \neq B}$, then we say that A is a **proper set** of A, also written

$$\mathbf{A \subset B.}$$

6. A set with no elements is called the **empty set** or **null set**, and is denoted by

$$\emptyset$$

7. The **Universal set**, U , is the set of all things to be considered in this case.

8. The **complement** of a set A , denoted by

$$A^{\circ} \text{ or } A' \text{ or } A^c$$

is the set of elements that belongs to U but which do not belong to A .

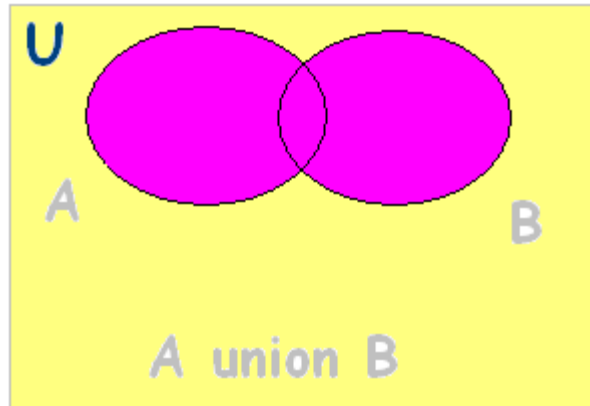
9. The **difference** between A and B , denoted by

$$A \setminus B$$

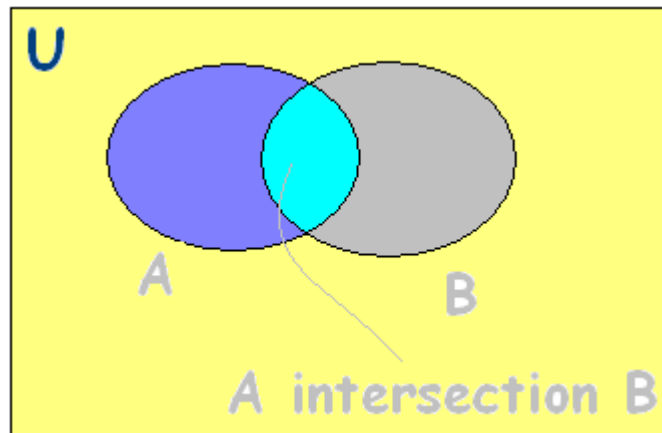
is the set of elements which belong to A but which do not belong to B .

10. The **symmetric difference** of the sets A and B , denoted by $A \oplus B$, consist of those elements which belong to A or B , but not both.

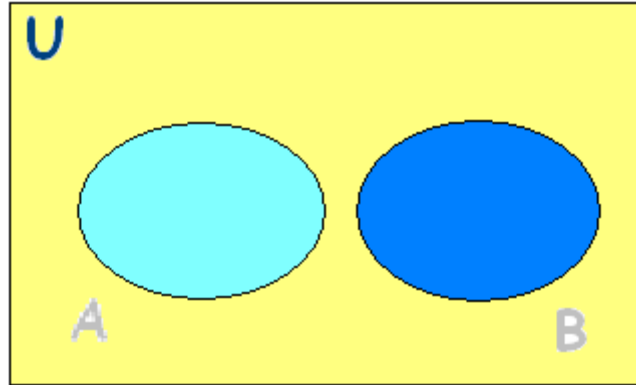
11. The **union** of two sets A and B , denoted by $A \cup B$, is the set of all elements which belong to A or to B .



12. The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of all elements which belong to both A and B.



13. Two A and B are said to be **disjoint** if they have no elements in common.



Addition Principle

Counting elements in a set:

$n(A)$ = number of elements in A (distinct)

$$n(\{1,2,3\})=4$$

F = set of female students in this room

M= set of male students in this room

$M \cup F$ set of students in this room

$$n(M \cup F) = n(M) + n(F)$$

A = set of business majors registered for class

B = set of students present today

$A \cup B$ = set of business majors or students present today

$A \cap B$ = set of business majors present today

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Addition Principle for Counting

For any two sets A and B

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

If A and B are disjoint, i.e. $A \cap B = \emptyset$ or $\{\}$

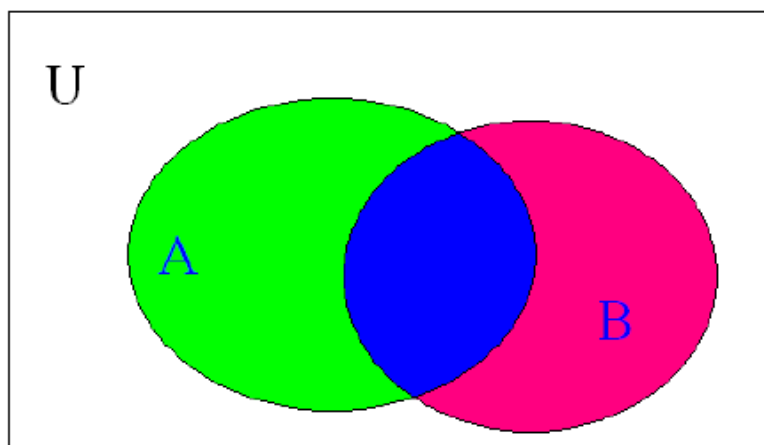
$$n(A \cup B) = n(A) + n(B)$$

Practical Problem

U students in this class 40
A students owning car
B students owning computer

30 students own a car
20 students own a computer
15 own both

How many have neither?



Assigning Students Numbers

Suppose each student is assigned a 5 digit number. How many different numbers can be created?

Each digit is 0,1,2,3,4,5,6,7,8,9

Ten possibilities for each

$$10 * 10 * 10 * 10 * 10 = 10^5 = 100,000$$

More interestingly

A) Suppose not digit is repeated

First slot: 10 Possibilities

Second 9 Possibilities

Third 8 Possibilities

Fourth 7 Possibilities

Fifth 6 Possibilities

$$\text{TOTAL} = 10 * 9 * 8 * 7 * 6 = 30240$$

B) No two adjacent digits are equal

First slot: 10 Possibilities

2,3,4,5: 9 Possibilities

$$\text{TOTAL} = 10 * 9 * 9 * 9 * 9 = 65610$$

Multiplication Rule

1. If two operations O_1 and O_2 are performed in order, with N_1 and N_2 possible outcomes, respectively, then there are

**$N_1 * N_2$
possible combined outcomes.**

2. For O_1, O_2, \dots, O_k operations with N_1, N_2, \dots, N_k possible outcomes, then there are

**$N_1 * N_2 * \dots * N_k$
possible combined outcomes**

Permutations and Combinations

_ FACTORIAL

_ PERMUTATIONS

_ COMBINATIONS

Factorial

Counting without replacement involved

$$25 * 24 * 23 * \dots * 2 * 1$$

and similar products, multiplying several numbers, each one less than the previous.

Beginning Lotto:

Pick 6 numbered balls out of 49 (numbered consecutively 1,2,3,...,49) without replacing them. How many ways can this be done (the order is important here)

First ball: 49 choices

Second : 48

Third: 47

Fourth: 46

Fifth: 45

Sixth: 44

Total: $49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 = 10,068,347,520$

So there are over 10 billion ways to pick the six balls. But the order is not important when playing Lotto.

1 2 3 4 5 6

6 5 4 3 2 1

1 3 2 4 5 6 etc. are the same

and should not be counted separate.

We will get back to that soon.

Factorial

For a natural number n ,

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$$

$$0! = 1$$

$$n! = n \cdot (n-1)!$$

Read “ n factorial”.

Examples:

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$7! = 7 \cdot (7-1)! = 7 \cdot (6!) = 7 \cdot 720 = 5040$$

$$\frac{49!}{43!} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot \dots \cdot 2 \cdot 1}{43 \cdot 42 \cdot \dots \cdot 2 \cdot 1}$$

$$= \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot (43!)}{43!} = 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$$

$$\frac{49!}{43!6!} = \frac{49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44}{6!} = \frac{10068347520}{720} = 13,983,816$$

Permutations

Recall the lotto example:
Order was not important!

Question:
How many ways can we **rearrange** or **permute** the numbers

1 2 3 4 5 6

We can choose any one of the 6 for the first slot, any of the remaining 5 for the second etc.

1st	6
2nd	5
3rd	4
4th	3
5th	2
6th	1

Total $6!=720$.

There is nothing particular about the numbers 1 2 3 4 5 6, any six distinct objects can be arranged in $6!$ different ways.

A Permutation of a Set of Objects

A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.

Number of Permutations of n Objects

The number of permutations of n distinct objects without repetition, denoted $P_{n,n}$, is given by

$$P_{n,n} = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 = n!$$

A permutation of n Objects r at a Time

In our Lotto example we have 49 numbers available but pick only 6 at a time:

49 choices for the first

48 for the second

...

44 for the sixth

$$49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 = \frac{49!}{43!} = \frac{49!}{(49-6)!} = P_{49,6}$$

A Permutation of n Objects, r at a time

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of the r objects in a specific order

The number of Permutations of n Objects taken r at a time

The number of permutations of n distinct objects taken r at a time without repetition is given by

$$\mathbf{P_{n,r} = n \cdot (n-1) \cdot \dots \cdot (n-r+1) \quad r \text{ factors}}$$

$$\mathbf{P_{n,r} = \frac{n!}{(n-r)!} \quad 0 \leq r \leq n \quad \mathbf{P_{n,n} = n!}}$$

Combinations

In our Lotto example the order of the six numbers did not matter. So we can NOT use the permutation formula.

Combination of n Objects taken r at a time

A combination of a set of n distinct objects taken r at a time without repetition is an r element subset of a set of n objects. The arrangement of the elements does NOT matter.

Simple Example:

How many ways can 2 out of 3 paintings of an artist be selected for shipment to an exhibition?
Let the paintings correspond to $\{A,B,C\}$

First choice:	A	B	C
Second:	B or C	A or C	A or B
Ordered:	(A,B) ; (A,C)	(B,A) ; (B,C)	(C,A) ; (C,B)

There are 6 permutations, but only 3 combinations: $\{A,B\};\{A,C\};\{B,C\}$

Number of Combinations of n objects taken r at a time

The number of combinations of n distinct objects taken r at a time without repetition is given by

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!} \quad 0 \leq r \leq n$$

How many 5 card hands can be drawn from a 52 card deck?

$$C_{52,5} = \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{120 \cdot (47!)} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{120} = 2,598,960$$

Permutation: ORDER MATTERS

Combination: ORDER DOES NOT MATTER